Newton-Cartan Connections with Torsion

Tekin Dereli*
Department of Physics, Koç University
34450 Sarıyer, İstanbul, Turkey

Şahin Koçak† and Murat Limoncu‡
Department of Mathematics, Anadolu University
26470 Eskişehir, Turkey

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Abstract

We re-formulate the notion of a Newton-Cartan manifold and clarify the compatibility conditions of a connection with torsion with the Newton-Cartan structure.

1 Introduction

According to Einstein's theory of general relativity, the space-time (M, g, ∇) is a four dimensional differentiable manifold M equipped with a type-(0,2), symmetric, non-degenerate, Lorentzian signatured metric tensor g and the unique metric-compatible, torsion-free Levi-Civita connection ∇. Elie Cartan [1] was first to notice that Newton's theory of gravitation can be formulated in a 4-dimensional space-time with a corank-1 degenerate metric [2], [3], [4]. The physical significance of Newton-Cartan space-times had been recognised after the 1960's [5], [6], [7]. It is well known that Cartan is also responsible for the concept of space-time torsion. In what follows we clarify the notion of Newton-Cartan manifolds and connections with torsion. In our main theorem (Th.2) we characterize n-dimensional Newton-Cartan space-times with torsion.

*E.mail: tdereli@ku.edu.tr
†E.mail: skocak@anadolu.edu.tr
‡E.mail: mlimoncu@anadolu.edu.tr

2 Newton-Cartan connections

Definition: 1 Let M be a smooth n-manifold, g and τ be (smooth) tensor fields of type $(0,2)$ and $(0,1)$, respectively, on M satisfying the following properties:
(G1) g is symmetric and degenerate with $rank(g) = n-1$,
(G2) There exists a vector field $V \in \mathcal{X}(M)$ such that, $g(V,X) = 0$ for all $X \in \mathcal{X}(M)$ and $\tau(V) = 1$.
We call such a triplet (M, g, τ) a Newton-Cartan manifold.

Remark: The vector field V in (G2) which is called the time vector field is determined up to a scalar function: If $g_p(V_p, X_p) = 0$ and $g_p(W_p, X_p) = 0$ for all $X_p \in T_pM$ $(p \in M)$ and for two vectors $V_p, W_p \in T_pM$, then V_p and W_p must be linearly dependent. Otherwise, V_p and W_p could be extended to a basis of T_pM, giving $rank(g) \leq n-2$ and thus contradicting (G1).

Convention: Given any non-degenerate $(0,2)$-tensor $\tilde{g}$ on a finite dimensional vector space V, a vector $X \in V$ determines a 1-form X^*, called the $\tilde{g}$-dual of X, by $X^*(Y) = \tilde{g}(X,Y)$ for any $Y \in V$. Likewise, a 1-form $\alpha \in V^*$ determines a vector α^*, called the $\tilde{g}$-dual of α, by $\tilde{g}(\alpha^*, Y) = \alpha(Y)$. We have $(X^*)^* = X$ and $(\alpha^*)^* = \alpha$. We will be using also the dual base $(f^1, \ldots, f^n)$ of V^* corresponding to a base $(e_1, \ldots, e_n)$ of V, given by $f^i(e_j) = \delta^i_j$, which is to be distinguished from the $\tilde{g}$-dual base $(e_1^*, \ldots, e_n^*)$.

We will consider a fixed vector field V with $g(V,X) = 0$ and $\tau(V) = 1$ also as part of a Newton-Cartan manifold. g restricted to the kernel of τ (more precisely to $ker\tau \times ker\tau$) is non-degenerate and one easily obtains:
Proposition:1 The tensor field $\bar{g}_\epsilon = g + \epsilon\tau \otimes \tau$, where ϵ is a non-zero real number, is symmetric and non-degenerate. Thus it is a semi-Riemannian metric on M. The dual τ^* of τ with respect to $\bar{g}_\epsilon$ is $\frac{1}{\epsilon}V$, i.e. $\tau(X) = \bar{g}_\epsilon(\frac{1}{\epsilon}V, X)$.
Remark: In the following, unless necesary, we will drop the subscript ϵ and understand that a fixed ϵ is chosen. We will stress the role of ϵ in Theorems 1 and 2.

Let us define $\bar{h}(\alpha, \beta) = \bar{g}(\alpha^*, \beta^*)$, where α and β are 1-forms and α^*, β^* their $\bar{g}$-duals. The $(2,0)$-tensor field $\bar{h}$ is symmetric and non-degenerate.

Proposition: 2 The $(2,0)$-tensor field $h = \bar{h} - \frac{1}{\epsilon}V \otimes V$ is symmetric and degenerate.

Proof: h is obviously symmetric. For degeneracy, let θ be any 1-form field. Then

$$\begin{aligned} h(\tau,\theta) &= \bar{h}(\tau,\theta) - \frac{1}{\epsilon}V(\tau)V(\theta) \\ &= \bar{g}(\tau^*,\theta^*) - \frac{1}{\epsilon}\tau(V)\theta(V) \\ &= \bar{g}(\frac{1}{\epsilon}V,\theta^*) - \frac{1}{\epsilon}\theta(V) \\ &= \frac{1}{\epsilon}(\bar{g}(V,\theta^*) - \theta(V)) = 0. \end{aligned}$$

Proposition:3 The following relations between $\bar{h}$ and $\bar{g}$ (resp. h and g) hold:

$$c_1^2(\bar{h}\otimes\bar{g}) = \delta\,,$$

$$c_1^2(h\otimes g) = \delta - V\otimes\tau$$

where ${c_1}^2$ denotes the $(2,1)$-contraction of the $(2,2)$-tensor $\bar{h}\otimes\bar{g}$ (resp. $h\otimes g$) and δ is the $(1,1)$-tensor $\delta(\alpha,X)=\alpha(X)$. In a local coordinate chart $(x^1,\dots,x^n)$ we have in terms of components

$$\sum_k \bar{h}^{ik}\bar{g}_{kj} = \delta^i_j,$$

$$\sum_k h^{ik}g_{kj} = \delta^i_j - V^i\tau_j.$$

These equalities are also true for any non-coordinate base $\{e_i\}$ and its dual base $\{f^j\}$ with $f^j(e_i)=\delta^j_i$.

Proof: Let $(\partial_1,\dots,\partial_n)$ denote the coordinate base, $(dx^1,\dots,dx^n)$ the dual base. Since $\bar{g}(\partial_i,\partial_j) = \bar{g}_{ij}$ and $\bar{h}(dx^i,dx^j) = \bar{h}^{ij}$ we have by definition $\sum_k \bar{h}^{ik}\bar{g}_{kj} = \delta^i_j$. Inserting in this $\bar{h}^{ik} = h^{ik} + \frac{1}{\epsilon}\tau^i\tau^k$ and $\bar{g}_{kj} = g_{kj} + \epsilon V_k V_j$ and taking into account $\tau(V) = 1$, $h(dx^i,\tau) = 0$, $g(V,\partial_j) = 0$ we get $\sum_k h^{ik}g_{kj} = \delta^i_j - \tau^i V_j$.

Having defined a semi-Riemannian manifold $(M,\bar{g})$ associated with the Newton-Cartan manifold (M,g,τ), we have the standard theory of connections at our disposal. Imposing certain conditions on connections on M, we will single out *Newton-Cartan* connections on (M,g,τ) compatible with g and τ in the following sense:

Definition: 2 Let ∇_X denote the covariant derivative of any linear connection $\mathcal{D}$ on the manifold M. We will call the connection $\mathcal{D}$ a *Newton-Cartan* connection iff

$$\nabla_X \tau = 0 \quad , \quad \nabla_X g = 0 \quad \forall X \in \mathcal{X}(M)$$

Proposition: 4 Let ∇_X denote the covariant derivative of a *Newton-Cartan* connection on M. Then $\nabla_X V = 0$ and $\nabla_X h = 0$, where V is the time vector field and h the (2,0)-field associated with g (s. Prop 2). Moreover, $\nabla_X \bar{h} = 0$.
Proof: First we note that $\nabla_X \bar{g} = \nabla_X g + \epsilon \left(\nabla_X \tau \otimes \tau + \tau \otimes \nabla_X \tau \right) = 0$. Now, from Prop.1, $\epsilon\tau(Y) = \bar{g}(V, Y)$ for any $Y \in \mathcal{X}(M)$. Then

$$\begin{aligned} \nabla_X(\epsilon\tau(Y)) &= \nabla_X(\bar{g}(V, Y)) \\ \epsilon(\nabla_X \tau)(Y) + \epsilon\tau(\nabla_X Y) &= (\nabla_X \bar{g})(V, Y) + \bar{g}(\nabla_X V, Y) + \bar{g}(V, \nabla_X Y) \\ \epsilon\tau(\nabla_X Y) &= \bar{g}(\nabla_X V, Y) + \bar{g}(V, \nabla_X Y) \end{aligned}$$

by $\nabla_X \tau = 0$ and $\nabla_X \bar{g} = 0$. Since $\epsilon\tau(\nabla_X Y) = \bar{g}\left(V, \nabla_X Y\right)$ we get $\bar{g}(\nabla_X V, Y) = 0$, hence $\nabla_X V = 0$.
Now we consider $h = \bar{h} - \frac{1}{\epsilon} V \otimes V$

$$\begin{aligned} \nabla_X h &= \nabla_X \bar{h} - \frac{1}{\epsilon} V \otimes \nabla_X V - \frac{1}{\epsilon} \nabla_X V \otimes V \\ \nabla_X h &= \nabla_X \bar{h} \end{aligned}$$

So it is enough to show $\nabla_X \bar{h} = 0$. Let α, β be 1-form fields and α^*, β^* their $\bar{g}$-dual vector fields. Since by definition $\bar{h}(\alpha, \beta) = \bar{g}(\alpha^*, \beta^*)$,

$$(\nabla_X \bar{h})(\alpha, \beta) = \nabla_X(\bar{h}(\alpha, \beta)) - \bar{h}(\nabla_X \alpha, \beta) - \bar{h}(\alpha, \nabla_X \beta).$$

On the other hand we have $(\nabla_X \alpha)^* = \nabla_X(\alpha^*)$:

$$\begin{aligned} \bar{g}(\alpha^*, U) &= \alpha(U) \qquad U \in \mathcal{X}(M) \\ \nabla_X(\bar{g}(\alpha^*, U)) &= \nabla_X(\alpha(U)) \\ \bar{g}(\nabla_X \alpha^*, U) + \bar{g}(\alpha^*, \nabla_X U) &= (\nabla_X \alpha)(U) + \alpha(\nabla_X U) \\ \bar{g}(\nabla_X \alpha^*, U) &= (\nabla_X \alpha)(U) \end{aligned}$$

because $\bar{g}(\alpha^*, \nabla_X U) = \alpha(\nabla_X U)$. This means $(\nabla_X \alpha)^* = \nabla_X \alpha^*$. Inserting these correspondences above and from $\nabla_X \bar{g} = 0$ we obtain

$$\left(\nabla_X \bar{h}\right)(\alpha, \beta) = \nabla_X(\bar{g}(\alpha^*, \beta^*)) - \bar{g}(\nabla_X \alpha^*, \beta^*) - \bar{g}(\alpha^*, \nabla_X \beta^*) = 0.$$

Remark: The converse of this proposition is also true and a kind of duality emerges. One could define a *Newton-Cartan* manifold also as a triplet (M, h, V).

The following theorem clarifies the existence and uniqueness of torsion-free *Newton-Cartan* connections:

Theorem:1 Let (M, g, τ) be a *Newton-Cartan* manifold.

i) If $\mathcal{D}$ is a *Newton-Cartan* connection without torsion, then $\mathcal{D}$ is the Levi-Civita connection on $(M, \overline{g}_\varepsilon)$ for all ϵ $(\neq 0)$ and $L_V g = 0, d\tau = 0$ where L_V denotes the Lie derivative, V being the time vector field $(\tau(V) = 1)$.

ii) Let $\mathcal{D}$ denote the Levi-Civita connection on $(M, \overline{g}_\varepsilon)$ for a fixed ϵ $(\neq 0)$. Assume, $L_V g = 0$ and $d\tau = 0$. Then, $\mathcal{D}$ is a torsion-free *Newton-Cartan* connection on (M, g, τ).

Remark: Theorem 1 shows that, there can't be any torsion-free *Newton-Cartan* connection, unless $L_V g = 0$ and $d\tau = 0$. In case of $L_V g = 0$ and $d\tau = 0$, there is a unique *Newton-Cartan* connection without torsion; it is the Levi-Civita connection for any $\bar{g}_\epsilon$ (especially, Levi-Civita connections for $\bar{g}_\epsilon$ coincide for all $\epsilon \neq 0$).

Proof (of theorem 1):

i) The first assertion (that $\nabla_X \overline{g}_\varepsilon = 0$) is trivial. So we show $d\tau = 0$ and $L_V g = 0$.

$$\begin{aligned}(\nabla_X \tau)(Y) &= \nabla_X (\tau(Y)) - \tau(\nabla_X Y) = 0 \\ (\nabla_Y \tau)(X) &= \nabla_Y (\tau(X)) - \tau(\nabla_Y X) = 0\end{aligned}$$

$$\begin{aligned}X\tau(Y) - Y\tau(X) &= \tau(\nabla_X Y - \nabla_Y X) = \tau([X, Y]) \\ (d\tau)(X, Y) &= 0 \qquad d\tau = 0.\end{aligned}$$

Now we will show $L_V g = 0$:

$$\begin{aligned}(\nabla_X g)(Y, Z) &= \nabla_X (g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \\ &= Xg(Y, Z) - g([X, Y] + \nabla_Y X, Z) - g(Y, [X, Z] + \nabla_Z X) = 0 \\ &= Xg(Y, Z) - g(L_X Y, Z) - g(Y, L_X Z) - g(\nabla_Y X, Z) - g(Y, \nabla_Z X) = 0\end{aligned}$$

Take $X = V$:

$$Vg(Y, Z) - g(L_V Y, Z) - g(Y, L_V Z) - g(\nabla_Y V, Z) - g(Y, \nabla_Z V) = 0$$

By Prop.4, $\nabla_Y V = 0$ $\nabla_Z V = 0$ and thus

$$Vg(Y, Z) - g(L_V Y, Z) - g(Y, L_V Z) = 0 \Longrightarrow (L_V g)(Y, Z) = 0 (L_V g = 0).$$

ii) It is enough to show $\nabla_X \tau = 0$ as $\nabla_X g = 0$ is then a consequence of $\overline{g}_\epsilon = g + \epsilon \tau \otimes \tau$ and $\nabla_X \overline{g}_\epsilon = 0$. (We will write below $\overline{g}$ for $\overline{g}_\epsilon$ again)

Let $Y \in \mathcal{X}(M)$. We get

$$
\begin{aligned}
(\nabla_X \tau)(Y) &= \nabla_X(\tau(Y)) - \tau(\nabla_X Y) \\
(\nabla_X \tau)(Y) &= X\left(\overline{g}\left(\frac{1}{\epsilon}V, Y\right)\right) - \overline{g}\left(\frac{1}{\epsilon}V, \nabla_X Y\right) \\
\epsilon(\nabla_X \tau)(Y) &= X(\overline{g}(V, Y)) - \overline{g}(V, \nabla_X Y)
\end{aligned}
$$

since $\frac{1}{\epsilon}V$ is $\overline{g}$-dual of τ. From the Koszul formula

$$
\begin{aligned}
2\overline{g}(\nabla_X Y, V) &= X\overline{g}(Y, V) + Y\overline{g}(V, X) - V\overline{g}(X, Y) \\
&\quad -\overline{g}(X, [Y, V]) + \overline{g}(Y, [V, X]) + \overline{g}(V, [X, Y])
\end{aligned}
$$

we can write

$$
\begin{aligned}
\epsilon(\nabla_X \tau)(Y) = {} & \tfrac{1}{2}\{X\overline{g}(V, Y) - Y\overline{g}(V, X) - \overline{g}(V, [X, Y])\} \\
& +\tfrac{1}{2}\{V\overline{g}(X, Y) - \overline{g}(X, [V, Y]) - \overline{g}(Y, [V, X])\}.
\end{aligned} \tag{1}
$$

The first term vanishes because of $d\tau = 0$:

$$
\begin{aligned}
(d\tau)(X, Y) &= X(\tau(Y)) - Y(\tau(X)) - \tau([X, Y]) = 0 \\
&= X\left(\overline{g}\left(\frac{1}{\epsilon}V, Y\right)\right) - Y\left(\overline{g}\left(\frac{1}{\epsilon}V, X\right)\right) - \overline{g}\left(\frac{1}{\epsilon}V, [X, Y]\right) = 0.
\end{aligned}
$$

The second term in (1) vanishes because of $L_V \overline{g} = 0$: (we will instantly show that $L_V \overline{g} = 0$)

$$
\begin{aligned}
(L_V \overline{g})(X, Y) &= V\overline{g}(X, Y) - \overline{g}(L_V X, Y) - \overline{g}(X, L_V Y) = 0 \\
&= V\overline{g}(X, Y) - \overline{g}(X, [V, Y]) - \overline{g}(Y, [V, X]) = 0
\end{aligned}
$$

Now the missing last part of the proof: $L_V \overline{g} = 0$. First we show that $L_V \tau = 0$:

$$
\begin{aligned}
(L_V \tau)(X) &= L_V(\tau(X)) - \tau(L_V X) \\
&= V(\tau(X)) - \tau([V, X]) \\
&= X(\tau(V)) \qquad (d\tau = 0) \\
&= 0 \qquad\qquad\quad (\tau(V) = 1)
\end{aligned}
$$

Now,

$$
\begin{aligned}
L_V \overline{g} &= L_V g + \epsilon L_V(\tau \otimes \tau) \\
L_V \overline{g} &= L_V g + \epsilon L_V \tau \otimes \tau + \epsilon \tau \otimes L_V \tau \\
L_V \overline{g} &= L_V g = 0
\end{aligned}
$$

by assumption of the theorem.

We generalize to the cases with torsion and prove our main theorem below:
Theorem:2 Let (M, g, τ) be a *Newton-Cartan* manifold
i) If $\mathcal{D}$ is a *Newton-Cartan* connection with torsion T ($T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$) then $\mathcal{D}$ is $\overline{g}_\epsilon$ compatible for all ϵ ($\neq 0$) (i.e. $\nabla_X \overline{g}_\epsilon = 0$) and

$$\begin{array}{l} (L_V g)(X,Y) = g(T(V,X),Y) + g(X,T(V,Y)) \\ (d\tau)(X,Y) = \tau(T(X,Y)) \end{array} \tag{2}$$

for all $X, Y, Z \in \mathcal{X}(M)$.
ii) Let $\mathcal{D}$ denote the connection on $(M, \overline{g}_\epsilon)$, compatible with $\overline{g}_\epsilon$ and with torsion T (for a fixed $\epsilon \neq 0$). Assume

$$\begin{array}{l} (L_V g)(X,Y) = g(T(V,X),Y) + g(X,T(V,Y)) \\ (d\tau)(X,Y) = \tau(T(X,Y)) \end{array}$$

for all $X, Y, Z \in \mathcal{X}(M)$. Then $\mathcal{D}$ is a *Newton-Cartan* connection on (M, g, τ) with torsion T.
Remark: We call the conditions (2) *compatibility conditions* of a torsion with the *Newton-Cartan* structure. Theorem 2 shows that, a *Newton-Cartan* connection with torsion T can exist, only if the torsion T is compatible with the *Newton-Cartan* structure. On the other hand, if T is compatible with the *Newton-Cartan* structure, then there exists a unique *Newton-Cartan* connection with torsion T; it is the connection compatible with any $\overline{g}_\varepsilon$ and having torsion T (especially, connections compatible with $\overline{g}_\varepsilon$ and having torsion T coincide for all $\epsilon \neq 0$).

Proof (of theorem 2)

i) The first assertion (that $\nabla_X \overline{g}_\epsilon = 0$) is again trivial. So we show the other two identities. First we show $(d\tau)(X,Y) = \tau(T(X,Y))$:

$$\begin{aligned} (\nabla_X \tau)(Y) &= \nabla_X (\tau(Y)) - \tau(\nabla_X Y) = 0 \\ (\nabla_Y \tau)(X) &= \nabla_Y (\tau(X)) - \tau(\nabla_Y X) = 0 \end{aligned}$$

$$\begin{aligned} X\tau(Y) - Y\tau(X) &= \tau(\nabla_X Y - \nabla_Y X) = \tau(T(X,Y) + [X,Y]) \\ (d\tau)(X,Y) &= X\tau(Y) - Y\tau(X) - \tau([X,Y]) = \tau(T(X,Y)). \end{aligned}$$

Now we show the identity about $L_V g$:

$$(\nabla_X g)(Y,Z) = \nabla_X (g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

$$Xg\left(Y,Z\right)-g\left(T\left(X,Y\right)+\left[X,Y\right]+\nabla_{Y}X,Z\right)-g\left(Y,T\left(X,Z\right)+\left[X,Z\right]+\nabla_{Z}X\right)=0$$

$$\begin{array}{l} Xg\left(Y,Z\right)-g\left(L_{X}Y,Z\right)-g\left(Y,L_{X}Z\right) \\ -g\left(T\left(X,Y\right)+\nabla_{Y}X,Z\right)-g\left(Y,T\left(X,Z\right)+\nabla_{Z}X\right)=0 \end{array}$$

Take $X=V$:

$$Vg\left(Y,Z\right)-g\left(L_{V}Y,Z\right)-g\left(Y,L_{V}Z\right)-g\left(T\left(V,Y\right)+\nabla_{Y}V,Z\right)-g\left(Y,T\left(V,Z\right)+\nabla_{Z}V\right)=0$$

By Prop.4, $\nabla_{Y}V=0$ $\nabla_{Z}V=0$ and thus

$$\begin{aligned} Vg\left(Y,Z\right)-g\left(L_{V}Y,Z\right)-g\left(Y,L_{V}Z\right) &= g\left(T\left(V,Y\right),Z\right)+g\left(Y,T\left(V,Z\right)\right) \\ (L_{V}g)(Y,Z) &= g\left(T\left(V,Y\right),Z\right)+g\left(Y,T\left(V,Z\right)\right). \end{aligned}$$

ii) Again, it is enough to show $\nabla_{X}\tau=0$ as $\nabla_{X}g=0$ follows from $\overline{g}=g+\epsilon\tau\otimes\tau$ and $\nabla_{X}\overline{g}=0$.

The equality for $\nabla_{X}\tau$,

$$\epsilon\left(\nabla_{X}\tau\right)\left(Y\right)=X\left(\overline{g}\left(V,Y\right)\right)-\overline{g}\left(V,\nabla_{X}Y\right)$$

is still true. But now we have the Koszul formula with torsion:

$$\begin{aligned} 2\overline{g}\left(\nabla_{X}Y,V\right) &= X\overline{g}\left(Y,V\right)+Y\overline{g}\left(V,X\right)-V\overline{g}\left(X,Y\right) \\ &\quad -\overline{g}\left(X,\left[Y,V\right]\right)+\overline{g}\left(Y,\left[V,X\right]\right)+\overline{g}\left(V,\left[X,Y\right]\right) \\ &\quad -\overline{g}\left(X,T\left(Y,V\right)\right)+\overline{g}\left(Y,T\left(V,X\right)\right)+\overline{g}\left(V,T\left(X,Y\right)\right) \end{aligned}$$

where $T\left(X,Y\right)=\nabla_{X}Y-\nabla_{Y}X-\left[X,Y\right]$.

Inserting this into the equality for $\nabla_{X}\tau$ we get

$$\begin{aligned} 2\epsilon\left(\nabla_{X}\tau\right)\left(Y\right) = &\{X\overline{g}\left(V,Y\right)-Y\overline{g}\left(V,X\right)-\overline{g}\left(V,\left[X,Y\right]\right)\} \\ &+\{V\overline{g}\left(X,Y\right)-\overline{g}\left(X,\left[V,Y\right]\right)-\overline{g}\left(Y,\left[V,X\right]\right)\} \\ &+\{\overline{g}\left(X,T\left(Y,V\right)\right)-\overline{g}\left(Y,T\left(V,X\right)\right)-\overline{g}\left(V,T\left(X,Y\right)\right)\}. \end{aligned} \tag{3}$$

We first simplify the first term in (3):

By assumption $\left(d\tau\right)\left(X,Y\right)=\tau\left(T\left(X,Y\right)\right)$

$$\begin{aligned} X\left(\tau\left(Y\right)\right)-Y\left(\tau\left(X\right)\right)-\tau\left(\left[X,Y\right]\right) &= \tau\left(T\left(X,Y\right)\right) \\ X\left(\overline{g}\left(\frac{1}{\epsilon}V,Y\right)\right)-Y\left(\overline{g}\left(\frac{1}{\epsilon}V,X\right)\right)-\overline{g}\left(\frac{1}{\epsilon}V,\left[X,Y\right]\right) &= \overline{g}\left(\frac{1}{\epsilon}V,T\left(X,Y\right)\right) \end{aligned}$$

Thus, the first term in (3) equals $\overline{g}\left(V,T\left(X,Y\right)\right)$.

To simplify the second term in (3) we use as before,

$$\begin{aligned}\left(L_V\overline{g}\right)(X,Y) &= V\overline{g}(X,Y)-\overline{g}(L_VX,Y)-\overline{g}(X,L_VY)\\ &= V\overline{g}(X,Y)-\overline{g}(X,[V,Y])-\overline{g}(Y,[V,X])\end{aligned}$$

Thus, the second term in (3) equals $\left(L_V\overline{g}\right)(X,Y)$. Let us compute $L_V\overline{g}$: Since $\overline{g}=g+\epsilon\tau\otimes\tau$

$$L_V\overline{g}=L_Vg+\epsilon L_V\tau\otimes\tau+\epsilon\tau\otimes L_V\tau$$

so we need $L_V\tau$:

$$\begin{aligned}\left(L_V\tau\right)(X) &= L_V\left(\tau(X)\right)-\tau\left(L_VX\right)\\ &= V\left(\tau(X)\right)-\tau\left([V,X]\right)\\ &= X\left(\tau(V)\right)+\tau\left(T(V,X)\right) && \left(d\tau\right)(V,X)=\tau\left(T(V,X)\right))\\ &= \tau\left(T(V,X)\right) && \left(\tau(V)=1\right)\end{aligned}$$

Inserting this in the formula for $L_V\overline{g}$, we obtain

$$\begin{aligned}\left(L_V\overline{g}\right)(X,Y) &= \left(L_Vg\right)(X,Y)+\epsilon\left(L_V\tau\right)(X)\,\tau(Y)+\epsilon\tau(X)\left(L_V\tau\right)(Y)\\ &= g(T(V,X),Y)+g(X,T(V,Y))\\ &\quad+\epsilon\tau\left(T(V,X)\right)\tau(Y)+\epsilon\tau(X)\,\tau\left(T(V,Y)\right)\\ &= \overline{g}(T(V,X),Y)+\overline{g}(X,T(V,Y))\end{aligned}$$

If we now replace the first term in (3) by $\overline{g}(V,T(X,Y))$ and second term in (3) by $\overline{g}(T(V,X),Y)+\overline{g}(X,T(V,Y))$ we get $\nabla_X\tau=0$ (by anti-symmetry of T).

Remark: There are two cases where the conditions of Theorem 2 simplify. Namely,

i. If $T(X,Y)\in ker\tau$, then the condition $(d\tau)(X,Y)=\tau(T(X,Y))$ reduces to $d\tau=0$.

ii. If $T(X,Y)=d\tau(X,Y)V$ then the condition $(d\tau)(X,Y)=\tau(T(X,Y))$ is satisfied. Because, $\tau(T(X,Y))=\tau(d\tau(X,Y)V)=d\tau(X,Y)\tau(V)=d\tau(X,Y)$ by $\tau(V)=1$. In this case, the condition $(L_Vg)(X,Y)=g(T(V,X),Y)+g(X,T(V,Y))$ reduces to $L_Vg=0$ because $g(T(V,X),Y)=g(d\tau(V,X)V,Y)=d\tau(V,X)g(V,Y)=0$ and similarly $g(X,T(V,Y))=0$.

Emerging from the considerations in Theorem 1 and Theorem 2 (and remarks following them), there holds the following characterization of *Newton-Cartan* connections, which is interesting by itself.

Proposition 5: A connection $\mathcal{D}$ (with or without torsion) on a *Newton-Cartan* manifold (M, g, τ) is a *Newton-Cartan* connection, if and only if it is compatible with all metrics $\overline{g}_\epsilon = g + \epsilon\tau \otimes \tau$ $(\epsilon \neq 0)$.
(i.e. $\nabla_X g = 0$ and $\nabla_X \tau = 0$ for all $X \in \mathcal{X}(M) \Longleftrightarrow \nabla_X \overline{g}_\epsilon = 0$ for all $\epsilon \neq 0$ and $X \in \mathcal{X}(M)$)
Proof: It is obvious that $\nabla_X g = 0$ and $\nabla_X \tau = 0$ imply $\nabla_X \overline{g}_\epsilon = 0$. Now assume $\nabla_X \overline{g}_\epsilon = 0$ for all $\epsilon \neq 0$. Indeed, it will suffice to assume $\nabla_X \overline{g}_{\epsilon_1} = 0$ and $\nabla_X \overline{g}_{\epsilon_2} = 0$ for two specific and distinct $\epsilon_1 \neq \epsilon_2$ (e.g. ± 1). From $\nabla_X g + \epsilon_1 \nabla_X (\tau \otimes \tau) = 0$ and $\nabla_X g + \epsilon_2 \nabla_X (\tau \otimes \tau) = 0$ we get $\nabla_X g = 0$ and $\nabla_X (\tau \otimes \tau) = 0$. By the product rule

$$\begin{aligned}(\nabla_X (\tau \otimes \tau))(Y, Z) &= \nabla_X ((\tau \otimes \tau)(Y, Z)) - (\tau \otimes \tau)(\nabla_X Y, Z) - (\tau \otimes \tau)(Y, \nabla_X Z) = 0 \\ &= X(\tau(Y)\tau(Z)) - \tau(\nabla_X Y)\tau(Z) - \tau(Y)\tau(\nabla_X Z) = 0\end{aligned}$$

for all $X, Y, Z \in \mathcal{X}(M)$. Taking $Y = Z = V$ we find $\tau(\nabla_X V) = 0$ by $\tau(V) = 1$. If we now take in the above identity only $Z = V$ and use $\tau(\nabla_X V) = 0$ we get

$$X(\tau(Y)) - \tau(\nabla_X Y) = 0$$

which means $(\nabla_X \tau)(Y) = 0$, i.e. $\nabla_X \tau = 0$.

3 Conclusion

We discuss here the relation between n-dimensional pseudo-Riemannian manifolds with parallel vector fields and Newton-Cartan structures. The latter consist of two parallel tensors, namely, a type-$(0, 2)$ symmetric tensor g that is degenerate of corank-(n-1) and a non-vanishing 1-form τ such that g is non-degenerate on $ker(\tau)$ and a compatible connection ∇. From this data, it is possible to construct a 1-parameter family of pseudo-Riemannian metrics $\bar{g}_\epsilon = g + \epsilon\tau \otimes \tau$ whose Levi-Civita connection coincide for any ϵ with the (torsion-free) Newton-Cartan connection and the 1-form τ and its dual vector V are parallel. Conversely, from a pseudo-Riemannian metric and a parallel vector field V one may reconstruct the Newton-Cartan structure making use of the dual 1-form τ. We give formulas that characterize a metric g and a parallel vector field V that are compatible with a Newton-Cartan connection with torsion. Most of these results were previously announced in an unpublished thesis [8].

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